ON EXISTENCE OF SOLUTION FOR IMPULSIVE PERTURBED QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS AND THE ASSOCIATED KURZWEIL EQUATIONS

S. A. BISHOP & O. O. AGBOOLA

Department of Mathematics Covenant University, Ota, Ogun State, Nigeria

ABSTRACT

Existence of solution of impulsive Lipschitzian quantum stochastic differential equations (QSDEs) associated with the Kurzweil equations are introduced and studied. This is accomplished within the framework of the Hudson-Parthasarathy formulation of quantum stochastic calculus and the associated Kurzweil equations. Here again, the solutions of a QSDE are functions of bounded variation, that is they have the same properties as the Kurzweil equations associated with QSDEs introduced in [1, 4]. This generalizes similar results for classical initial value problems to the noncommutative quantum setting.

KEYWORDS: Impulsive, Kurzweil Equations, Bounded Variation, Noncommutative Stochastic Processes

INTRODUCTION

Impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as biology, medicine, economics, mechanics, electronics, Physics, etc [2, 3, 8, 9, 12, 13, 15, 18]. Thus the qualitative properties of the mathematical theory of impulsive differential systems are very important. A lot of dynamical systems have variable structure subject to stochastic abrupt changes, which may result from abrupt phenomena such as stochastic failures and repairs of components, sudden environmental changes, etc [8, 10, 20-22].

Recently, stochastic differential equations have attracted a great attention, since they have been used extensively in many areas of application including finance and social science [8-10, 12, 13, 20-22] The existence, uniqueness and asymptotic behavior of solutions of stochastic differential equations have been considered by many authors [2, 3, 8, 9, 13]. However, within the framework of the Hudson and Parthasarathy [11] formulation of QSDE not much has been done. In [15] the existence of QSDE that exhibit impulsive effects was established using fixed point theorem.

In [1], the equivalence of the non classical ODE (QSDE) and the associated Kurzweil equation was established along side with some numerical examples. It is worth mentioning that the results in [1] have proved to be very efficient when compared with results obtained from other schemes. Again using this method in [4], we studied Measure quantum stochastic differential systems (systems that exhibit discontinuous solutions) with examples. We established such results by considering the associated Kurzweil equations [4, 9, 17]. The motivation for studying the existence of solution for impulsive QSDE associated with Kurzweil equations is so that we can subsequently use the method in [1] to obtain similar approximate results for this class of QSDEs.

In this paper we describe another approach to systems that exhibit impulsive behaviour. We rely on the formulations of [1] concerning the equivalent QSDE and the associated Kurzweil equation. The methods are simple extension of the methods applied in [14, 16-19] to this non commutative quantum setting involving unbounded linear

operators in locally convex spaces. Hence the results obtained here are generalizations of similar results obtained in [16, 17] concerning classical initial value problems. The rest of this paper is organized as follows.

In section 2 we present some definitions, preliminary results and notations. In section 3, we establish the main results.

All through the remaining sections, as in [1, 4, 6, 7] we employ the locally convex topological space $\tilde{A}$ of non commutative stochastic processes. We also adopt the definitions and notations of the following spaces Ad($\tilde{A}$), Ad($\tilde{A}$)$_{wac}$, $L_{loc}^p(\tilde{A})$, $L_{loc}^\infty(\mathbb{R}_+)$, C($\tilde{A}\times$[a, b], W), $\mathcal{F}(\tilde{A} \times$ [a, b], $h_{\eta\xi}$, W) and the integrator processes Λ_Π, A_f^+, A_g. We consider the quantum stochastic differential equation in integral form given by

$$X(t) = X_0 + \int_0^t \left(E(t, X(s)) d\Lambda_\pi(s) + F(s, X(s)) dA_f^+(s) + G(s, X(s)) dA_g(s) + H(s, X(s)) ds \right), t \in [t_0, T] \tag{1.1}$$

In equation (1.1), the coefficients E, F, G, and H lie in a certain class of stochastic processes for which quantum stochastic integrals against the gauge, creation, annihilation processes Λ_π, A_f^+, A_g and the Lebesgue measure t are defined in [7]. In the work of [7], the Hudson and Parthasarathy [11] quantum stochastic calculus was employed to establish the equivalent form of quantum stochastic differential equation (1.1) given by

$$\frac{d}{dt}\langle \eta, X(t)\xi \rangle = P(X(t), t)(\eta, \xi)$$

$$X(t_0) = X_0 , t \in [t_0, T] , \tag{1.2}$$

where η, ξ lie in some dense subspaces of some Hilbert spaces which has been defined in [7]. For the explicit form of the map $P(x, t) \to P(x, t)(\eta, \xi)$ appearing in equation (1.2), see [1, 7]. Equation (1.2) is a first order non-classical ordinary differential equation with a sesquilinear form valued map P as the right hand side. In [1], the equivalence of the non-classical ordinary differential equation (1.2) with the associated Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, X(\tau)\xi \rangle = DF(X(\tau), t)(\eta, \xi) , t \in [t_0, T] \tag{1.3}$$

was established along with some numerical examples. The map F in (1.3) is given by

$$F(x, t)(\eta, \xi) = \int_0^t P(x, s)(\eta, \xi) ds \tag{1.4}$$

2. NOTATIONS, DEFINITIONS AND PRELIMINARY RESULTS

We shall employ certain spaces of maps (introduced above) whose values are sesquilinear forms on ($\mathbb{D} \otimes \mathbb{E}$).

2.1 Definition

A member $z \in L^0(I, \mathbb{D} \otimes \mathbb{E}$) is:

- Absolutely continuous if the map t$\to z(t)(\eta, \xi)$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$

- of bounded variation if over all partition $\{t_j\}_{j=0}^n$ of I,

  $\text{Sup}_I(\sum_{j=1}^n |Z(t_j)(\eta, \xi) - Z(t_{j-1})(\eta, \xi)|) < \infty.$

- of essentially bounded variation if z is equal almost everywhere to some member of $L^0(I, \mathbb{D} \otimes \mathbb{E}$) of bounded variation.

- A stochastic process $X : [t_0, T] \longrightarrow \hat{\mathcal{A}}$ is of bounded variation if

On Existence of Solution for Impulsive Perturbed Quantum Stochastic
Differential Equations and the Associated Kurzweil Equations

53

$$\text{Sup} \left(\sum_{j=1}^{n} |\langle \eta, X(t_j)\xi \rangle - \langle \eta, X(t_{j-1})\xi \rangle| \right) < \infty.$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and where supremum is taken over all partitions

$\{t_j\}_{j=0}^{n}$ of I.

2.2 Notation

We denote by BV(Ã) the set of all stochastic processes of bounded variation on I.

2.3 Definition

For x ∈ BV(Ã), define for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$,

$$\text{Var}_{[a,b]}X_{\eta\xi} = \text{Sup}_{\tau} \left(\sum_{j=1}^{n} \|X(t_j) - X(t_{j-1})\|_{\eta\xi} \right)$$

where τ is the collection of all partitions of the interval [a, b] ⊂ I. If [a, b] = I, we simply write $\text{Var}_I X_{\eta\xi} = \text{Var}X_{\eta\xi}$. Then $\{\text{Var}X_{\eta\xi}, \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ is a family of seminorms which generates a locally convex topology on BV(Ã).

2.4 Notation

- We denote by $\overline{BV}(\tilde{A})$ the completion of BV(Ã) in the said topology.

- For any member Z of $L^0(I, \mathbb{D} \underline{\otimes} \mathbb{E})$ of bounded variation, we write $\text{Var}Z_{\eta\xi}$ for its variation on [a, b] ⊆ I.

- We denote by A:= BV(Ã)∩ Ad(Ã)$_{\text{wac}}$ the stochastic process that are weakly, absolutely continuous and of bounded variation on $[t_0,T]$.

- We denote by $C(\hat{\mathcal{A}} \times [t_0, T], W)$ the class of sesquilinear form – valued maps which are Lipschitzian and satisfy the Caratheodory conditions as defined below.

- We denote by $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ the class of sesquilinear form – valued maps that are Kurzweil integrable as defined below.

2.5 Definition

For each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ let $h_{\eta\xi}:[t_0, T] \longrightarrow \mathbb{R}$ be a family of non decreasing function defined on $[t_0$, T] and $W:[0, \infty) \longrightarrow \mathbb{R}$ be a continuous and increasing function such that $W(0) = 0$. Then we say that the map $F : \hat{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ belongs to the class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ if for all $x, y \in \hat{\mathcal{A}}, t_2 t_1 \in [t_0, T]$

- $|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi)| \leq |h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (2.1)

- $|F(x, t_2)(\eta, \xi) - F(x, t_1)(\eta, \xi) + F(y, t_2)(\eta, \xi) - F(y, t_1)(\eta, \xi)|$

  $\leq W(\|x - y\|_{\eta\xi})|h_{\eta\xi}(t_2) - h_{\eta\xi}(t_1)|$ (2.2)

Next we give a result that connects the class $C(\hat{\mathcal{A}} \times [t_0, T], W)$ with the class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$.

2.6 Definition

A map $P : \hat{\mathcal{A}} \times [t_0, T] \longrightarrow sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ is of the class $C(\hat{\mathcal{A}} \times [t_0, T], W)$ if for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

- $P(x, .)(\eta, \xi)$ is measurable for each $x \in \hat{\mathcal{A}}$.

- There exists a family of measurable functions $\tilde{M}_{\eta\xi}: [t_0, T] \to \mathbb{R}_+$ such that

$$\int_{t_0}^{T} M_{\eta\xi}(s)ds < \infty \text{ and } |P(x,s)(\eta,\xi)| \leq M_{\eta\xi}(s), (x,s) \in \hat{\mathcal{A}} \times [t_0, T] \tag{2.3}$$

- There exists a family of measurable functions $K_{\eta\xi}^p: [t_0, T] \to \mathbb{R}_+$ such that for each $t \in [t_0, T]$, $\int_{t_0}^{T} K_{\eta\xi}(s)ds < \infty$, and

$$|P(x,s)(\eta,\xi) - P(x,s)(\eta,\xi)| \leq K_{\eta\xi}^p(s)W(\|x-y\|_{\eta\xi}) \tag{2.4}$$

For $(x,s), (y,s) \in \hat{\mathcal{A}} \times [t_0, T]$ and all through $W(t) = t$.

MAJOR RESULTS

Assume that the set $A(\eta, \xi)$ defined in [6] is compact in $\mathbb{C}$, $[t_0, T] \subseteq I$. Let $P : \hat{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ satisfy the conditions (2.3) and (2.4). Further let a finite set of points $t_i \in [t_0, T]$, $i = 1, 2, \ldots, k$ be given with $t_i < t_{i+1}$ for $i = 1, 2, \ldots, k-1$ and a system of k continuous maps

$$z_i: A \to sesq(\mathbb{D} \underline{\otimes} \mathbb{E}), i = 1, 2, \ldots, k$$

The QSDE with impulsive action at the fixed points $t_1, t_2, \ldots, t_k$ is of the form

$$X(t) = X_0 + \int_0^t \left(E(t, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f^+(s) + G(s, X(s))dA_g(s) + H(s, X(s))ds \right)$$

$$+ \sum_{0 < t_i < t} z_i(x)H_{t_i}(t) , \ t \neq t_i \tag{3.1}$$

$$\Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-) = z_i(x(t_i)) \tag{3.2}$$

The equivalent form of (3.1) and (3.2) is given by

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(x,t)(\eta,\xi) + \sum_{0 < t_i < t} z_i(x_{\eta\xi})H_{\eta\xi,t_i}(t) , \ t \neq t_i \tag{3.3}$$

$$\langle \eta, \Delta x\xi \rangle|_{t=t_i} = \langle \eta, x(t_i^+)\xi \rangle - \langle \eta, x(t_i)\xi \rangle = \langle \eta, z_i(x(t_i))\xi \rangle \tag{3.4}$$

The equation (3.3) describes the behaviour of the state at the points different from $t_i, i = 1, 2, \ldots, k$ and (3.4) represents the discontinuity from the right of the solution for $t = t_i$ and satisfy the Lipschitz conditions defined in 2.6. Equation (3.3) is given in integral form as

$$\langle \eta, x(t)\xi \rangle - \langle \eta, x(0)\xi \rangle = \int_0^t P(x,s)(\eta,\xi)ds + \sum_{0 < s_i < s} z_i(x_{\eta\xi})H_{\eta\xi,s_i}(t)$$

The Kurzweil equation associated with equation (3.3) is given by

$$\frac{d}{dt}\langle \eta, x(\tau)\xi \rangle = D[F(x(\tau),t)(\eta,\xi) + \sum_{0 < t_i < t} z_i(x_{\eta\xi})H_{\eta\xi,t_i}(t)] , \ t \neq t_i \tag{3.5}$$

The differential system with impulses (3.3) and (3.4) is best described by its solution as follows:

3.1 Definition

A stochastic process $x: [a,b] \subset [t_0, T] \to \tilde{A}$ is called a solution of the quantum stochastic differential

On Existence of Solution for Impulsive Perturbed Quantum Stochastic
Differential Equations and the Associated Kurzweil Equations

55

equation (3.3) and (3.4) if $(x(t), t) \in A \times [t_0, T]$ for $t \in [a, b]$, $x \in Ad(\tilde{A})_{wac}$ on every interval $[t_0, t_1] \cap [a, b]$, $(t_i, t_{i+1}) \cap [a, b]$, $i = 1, 2, \ldots, k-1$, $(t_k, b] \cap [a, b]$ and

$$\langle \eta, x(t_2)\xi \rangle - \langle \eta, x(t_1)\xi \rangle = \int_0^t P(x, s)(\eta, \xi)ds + \sum_{0 < t_i < t} z_i(x_{\eta\xi}) H_{\eta\xi, t_i}(t), t_1, t_2 \in [t_0, T].$$

For a given $d \in [t_0, b)$ define $H_{\eta\xi, d}(t) = 0$ for $t \le d$, $H_{\eta\xi, d}(t) = 1$ for $t > d$.

Where the relationship between the maps P and F is as defined below in definition 3.2.

3.2 Definition

$$F(x, t)(\eta, \xi) = \int_0^t P(x, s)(\eta, \xi)ds + \sum_{0 < t_i < t} z_i(x_{\eta\xi}) H_{\eta\xi, t_i}(t), t_0 = 0 \tag{3.6}$$

Where $F : \hat{A} \times [t_0, T] \longrightarrow sesq(\mathbb{D} \underline{\otimes} \mathbb{E})$ belongs to the class $\mathcal{F}(\hat{A} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

The following result is a consequence of definition 3.1.

3.2 Theorem

A stochastic process : $[a, b] \to \tilde{A}$, $[a, b] \subset [t_0, T]$ is a solution of the nonclassical differential equation (3.3) with impulses (3.4) on $[a, b]$ if and only if x satisfies definition (3.1).

Proof

Assume that the stochastic process $x : [t_0, T] \to \tilde{A}$ is a solution of equation (3.3), then for $t \in (t_i, t_{i+1}]$,

$$\int_0^t P(x(s), s)(\eta, \xi)ds = \int_0^t \frac{d}{ds}\langle \eta, x(s)\xi \rangle ds$$

$$\int_0^{t_1} \frac{d}{ds}\langle \eta, x(s)\xi \rangle ds + \int_{t_1}^{t_2} \frac{d}{ds}\langle \eta, x(s)\xi \rangle ds + \cdots + \int_{t_i}^t \frac{d}{ds}\langle \eta, x(s)\xi \rangle ds$$

$$= [\langle \eta, x(t_1^-)\xi \rangle - \langle \eta, x(0^+)\xi \rangle] + [\langle \eta, x(t_2^-)\xi \rangle - \langle \eta, x(t_1^+)\xi \rangle] + \cdots +$$

$$+ [\langle \eta, x(t^-)\xi \rangle - \langle \eta, x(t_i^+)\xi \rangle]$$

$$= [\langle \eta, x(t_1^-)\xi \rangle - \langle \eta, x(0)\xi \rangle] + [\langle \eta, x(t_2^-)\xi \rangle - \langle \eta, x(t_1^+)\xi \rangle] + \cdots +$$

$$+ [\langle \eta, x(t)\xi \rangle - \langle \eta, x(t_i^+)\xi \rangle]$$

$$= \langle \eta, x(0)\xi \rangle - [\langle \eta, x(t_1^+)\xi \rangle - \langle \eta, x(t_1^-)\xi \rangle] - [\langle \eta, x(t_2^+)\xi \rangle - \langle \eta, x(t_2^-)\xi \rangle] - \cdots -$$

$$- [\langle \eta, x(t_i^+)\xi \rangle - \langle \eta, x(t_i^-)\xi \rangle] + \langle \eta, x(t)\xi \rangle$$

Hence

$$\langle \eta, x(t)\xi \rangle - \langle \eta, x(0)\xi \rangle = \int_0^t P(x(s), s)(\eta, \xi)ds + [\langle \eta, x(t_1^+)\xi \rangle - \langle \eta, x(t_1^-)\xi \rangle] +$$

$$+ [\langle \eta, x(t_2^+)\xi \rangle - \langle \eta, x(t_2^-)\xi \rangle] + \cdots + [\langle \eta, x(t_i^+)\xi \rangle - \langle \eta, x(t_i^-)\xi \rangle]$$

$$= \langle \eta, x(0)\xi \rangle + \int_0^t P(x(s), s)(\eta, \xi)ds + \sum_{0 < t_i < t} \Delta x_{\eta\xi}(t_i)$$

$$= \langle \eta, x(0)\xi \rangle + \int_0^t P(x(s), s)(\eta, \xi)ds + \sum_{0 < t_i < t} z_i(x_{\eta\xi}(t_i)) H_{\eta\xi, t_i}(t) \tag{3.7}$$

Conversely, if $x(.) \in A$ satisfies (3.7) for $t \in (t_i, t_{i+1})$, since $\sum_{0 < t_i < t} z_i(x_{\eta\xi}(t_i))$ is a constant and its derivative is zero for $t \neq t_i$, $i = 1, 2, \dots, k$. Hence, we deduce that

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(x,t)(\eta,\xi), t \neq t_i$$

$$\langle \eta, x(0)\xi \rangle = \langle \eta, x_0\xi \rangle, \text{and}$$

$$\langle \eta, x(t_i)\xi \rangle = \langle \eta, x(t_i^+)\xi \rangle - \langle \eta, x(t_i^-)\xi \rangle$$

$$= [\langle \eta, x(0)\xi \rangle + \int_0^{t_i} P(x(s),s)(\eta,\xi)ds + \sum_{i=1}^{k} z_i\left(x_{\eta\xi}(t_i)\right) H_{\eta\xi,t_i}(t)]$$

$$= [\langle \eta, x(0)\xi \rangle + \int_0^{t_i} P(x(s),s)(\eta,\xi)ds + \sum_{i=1}^{k-1} z_i\left(x_{\eta\xi}(t_i)\right) H_{\eta\xi,t_i}(t)]$$

$$= z_i\left(x_{\eta\xi}(t_i)\right) = \langle \eta, z_i(x(t_i))\xi \rangle.$$

We have the following results that connect the two classes of maps F and P together.

3.2 Theorem

For each $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$ Assume that the map $F : \hat{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D} \otimes \mathbb{E})$ belongs to the class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ and $P : \hat{\mathcal{A}} \times [t_0, T] \to sesq(\mathbb{D} \otimes \mathbb{E})$ belongs to the class $(\hat{\mathcal{A}} \times [t_0, T], W)$. Then for every $x, y \in \hat{\mathcal{A}}, t_2, t_1 \in [t_0, T]$, $F(x,t)(\eta,\xi)$ defined by (3.6) satisfies

- $|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| \leq \int_{t_1}^{t_2} M_{\eta\xi}^1(s)ds + C_{\eta\xi} \int_{t_1}^{t_2} M_{\eta\xi}^2(s)ds$

- $|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) + F(y,t_2)(\eta,\xi) - F(y,t_1)(\eta,\xi)|$

$$\leq W(\|x - y\|_{\eta\xi}) \int_{t_0}^{T} K_{\eta\xi}^p(s)ds$$

The map $F(x,t)(\eta,\xi)$ is of class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}, W)$ for each $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$, where

$$h_{\eta\xi}(t) = \int_{t_0}^{t} M_{\eta\xi}(s)ds + \int_{t_0}^{t} K_{\eta\xi}^p(s)ds$$

Proof

Since $A(\eta,\xi)$ is compact and the maps are continuous, there exists a constant $C_{\eta\xi} \geq 0$ such that $|\langle \eta, z_i(x)\xi \rangle| \leq C_{\eta\xi}$ for all $x_{\eta\xi} \in A(\eta,\xi)$ and $i = 1, 2, \dots, k$. Therefore since (2.3) holds we have by (3.6) and for all $x \in A(\eta,\xi), t_2, t_1 \in [t_0, T]$

$$|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi)| \leq$$

$$\leq \left|\int_{t_1}^{t_2} P(x,s)(\eta,\xi)ds\right| + C_{\eta\xi}\left|\sum_{i=1}^{k} H_{\eta\xi,t_i}(t_2) - H_{\eta\xi,t_i}(t_1)\right|$$

$$\leq \int_{t_1}^{t_2} M_{\eta\xi}^1(s)ds + C_{\eta\xi} \int_{t_1}^{t_2} M_{\eta\xi}^2(s)ds$$

$$\leq |h_{\eta\xi}^1(t_2) - h_{\eta\xi}^1(t_1)| + C_{\eta\xi}|h_{\eta\xi}^2(t_2) - h_{\eta\xi}^2(t_1)|$$

On Existence of Solution for Impulsive Perturbed Quantum Stochastic
Differential Equations and the Associated Kurzweil Equations

57

where $h_{\eta\xi}^1 : [t_0, b] \to \mathbb{R}$ is as defined in [4] where $F_1(x,t)(\eta,\xi) = \int_0^t P(x,s)(\eta,\xi)ds$ belongs to the class $\mathcal{F}(\hat{\mathcal{A}} \times [t_0, T], h_{\eta\xi}^1, W_1)$ and

$$h_{\eta\xi}^2(t) = \sum_{i=1}^k H_{\eta\xi, t_i}(t), t \in [t_0, T].$$

Clearly $h_{\eta\xi}^2$ is nondecreasing and continuous from the left on $[t_0, T]$. If W_2 is the common modulus of continuity of the finite systems of mappings $z_i, i = 1, 2, \dots, k$ then

$$\|z_i(x) - z_i(y)\|_{\eta\xi} \leq W_2(\|x - y\|_{\eta\xi})$$

for $x, y \in A$. Using the information from [1, 4] on the Caratheodory equations, we obtain

$$|F(x,t_2)(\eta,\xi) - F(x,t_1)(\eta,\xi) + F(y,t_2)(\eta,\xi) - F(y,t_1)(\eta,\xi)|$$

$$\leq W_1(\|x - y\|_{\eta\xi})|h_{\eta\xi}^1(t_2) - h_{\eta\xi}^1(t_1)| + W_2(\|x - y\|_{\eta\xi})|h_{\eta\xi}^2(t_2) - h_{\eta\xi}^2(t_1)|$$

for $x, y \in A(\eta, \xi)$ and $t_2, t_1 \in [t_0, T]$. The first term correspond to P and for the second term in (3.6) we have the following estimate

$$\left| \sum_{i=1}^k \left(z_i(x_{\eta\xi}) - z_i(y_{\eta\xi}) \right) (H_{\eta\xi, t_i}(t_2) - H_{\eta\xi, t_i}(t_1)) \right|$$

$$\leq W_2(\|x - y\|_{\eta\xi}) \sum_{i=1}^k (H_{\eta\xi, t_i}(t_2) - H_{\eta\xi, t_i}(t_1))$$

$$\leq W_2(\|x - y\|_{\eta\xi})|h_{\eta\xi}^2(t_2) - h_{\eta\xi}^2(t_1)|$$

where If we take $h_{\eta\xi}(t) = h_{\eta\xi}^1(t) + h_{\eta\xi}^2(t)$ for $t \in [t_0, T]$ and $W(r) = W_1(r) + W_2(r)$ then we obtain that the map $F(x,t)(\eta,\xi)$ defined by (3.6) belongs to the class $\mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$.

We now present the major result in this section.

3.3 Theorem

A stochastic process $: [a, b] \to \tilde{A}$, $[a, b] \subset [t_0, T]$ is a solution of the nonclassical differential equation (3.3) with impulses (3.4) on $[a, b]$ if and only if x is a solution of (3.5).

Proof

That a stochastic process $x: [a, b] \to \tilde{A}$ is a solution of the nonclassical differential equation (3.3) with impulses (3.4) on $[a, b]$. By theorems 3.2 and 4.4 in [1], the integral $\int_{t_1}^{t_2} DF(x(\tau), t)(\eta, \xi)$ exists and

$$\langle \eta, x(t_2)\xi \rangle - \langle \eta, x(t_1)\xi \rangle = \int_{t_1}^{t_2} P(x(t), t)(\eta, \xi)ds + \sum_{i=1}^{k-1} z_i(x_{\eta\xi}(t_i)) H_{\eta\xi, t_i}(t)$$

$$= \int_{t_1}^{t_2} D[F(x(\tau), t)(\eta, \xi) + \sum_{i=1}^{k-1} z_i(x_{\eta\xi}(t_i)) H_{\eta\xi, t_i}(t)] \tag{3.8}$$

for all $t_1, t_2 \in [a, b]$. Hence x is a solution of (3.5).

Conversely, if x is a solution of (3.5), then by theorem 3.2 x satisfies eq. (3.3). Since $F(x(\tau), t)(\eta, \xi)$ is of class $\mathcal{F}(\tilde{A} \times [a, b], h_{\eta\xi}, W)$, we have

$$\langle \eta, x(t_2)\xi \rangle - \langle \eta, x(t_1)\xi \rangle = \left| \int_{t_1}^{t_2} D[F(x(\tau), t)(\eta, \xi) + \sum_{i=1}^{k-1} z_i(x_{\eta\xi}(t_i)) H_{\eta\xi, t_i}(t)] \right|$$

$$\leq \left| h_{\eta\xi}^1(t_2) - h_{\eta\xi}^1(t_1) \right| + C_{\eta\xi} \left| h_{\eta\xi}^2(t_2) - h_{\eta\xi}^2(t_1) \right| \tag{3.9}$$

Hence by theorem 5.1 in [1], we have

$$\int_{t_1}^{t_2} P(x(t), t)(\eta, \xi)ds + \sum_{i=1}^{k-1} z_i(x_{\eta\xi}(t_i)) H_{\eta\xi, t_i}(t) = \int_{t_1}^{t_2} D[F(x(\tau), t)(\eta, \xi) + \sum_{i=1}^{k-1} z_i(x_{\eta\xi}(t_i)) H_{\eta\xi, t_i}(t)].$$

The theorem has established the fact that x is a solution of (3.5) if and only if (3.8) holds by equation (3.9). This follows if and only if equation (3.3) and (3.4) hold, and the theorem is established.

REMARK

The above result holds since the equivalence of the two equations have been established in [1]. This work would have applications in the theory of quantum continuous measurements and in areas such as mechanics, electrical engineering, medicine biology, and ecology.

CONCLUSIONS

We have established existence of solutions for a class of impulsive quantum stochastic differential equations associated with the Kurzweil equations, which generalizes analogous results due to the references [15, 17-19].

REFERENCES

1. Ayoola, E.O. (2001) Lipschitzian Quantum Stochastic Differential Equations and the Associated Kurzweil Equations, Stochastic analysis and applications, 19 (4), 581-603.

2. Ballinger, G, Liu, X. (1999) Existence and uniqueness results for impulsive delay differential equations, Dynam. contin. discrete impuls. systems (5) 579–591.

3. Benchohra, M, Henderson, J and Ntouyas, S.K. (2002), Existence results for impulsive semilinear neutral functional differential equations in Banach spaces, Memoirs on Diff. Equ. Math. Phys., 25, 105-120.

4. Bishop, S. A. (2012) 'Discontinuous quantum stochastic differential equations and the associated kurzweil equations', IJBAS, 12(4), 25-31.

5. Bishop, S. A and Ayoola, E. O. (2013)'Variational Stability for Kurzweil Equations associated with Quantum Stochastic Differential Equations', Australian J. of basic and applied sciences, May issue,787-798.

6. Da Prato, G and Zabczyk, J. (1992),'Stochastic equations in infinite dimensions', Vol 44 Encyclopedia of Mathematics and its Applications, Cambridge university Press, Cambridge, Mass, USA.

7. Ekhaguere, G. O. S. (1992), 'Lipschitzian Quantum Stochastic Differential Inclusions. Int. journal of theoretical physics'. 31(11), 2003 - 2034.

8. Guendouzi, T and Mehdi, K. (2013), 'Existence of mild solutions for impulsive fractional stochastic equations with infinite delay'. Malaya journal of matematik 4(1), 30–43.

On Existence of Solution for Impulsive Perturbed Quantum Stochastic
Differential Equations and the Associated Kurzweil Equations

59

9. Hern´andez, E, Pierri, M, and Goncalves, G.(2006). 'Existence results for an impulsive abstract partial Differential equation with state-dependent delay'. Computers & mathematics with applications, 52(3-4), 411- 420.

10. Hespanhaa, J. P., Daniel, L. and Andrew R. T. (2008), 'Lyapunov conditions for input-to-state stability of impulsive systems'. Automatica 44, 2735-2744.

11. Hudson, R. L. and Parthasarathy, K. R. (1984), 'Quantum Ito's formulae and stochastic evolutions'. Comm. math. phys., 93, 301-324.

12. Lakshmikantham, V, Baˇınov, D and Simeonov, P. (1989), 'Theory of Impulsive Differential Equations', World scientific publishing.

13. Lin, A and Hu, L. (2010), 'Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions', Computers & mathematics with applications, 59, 64-73.

14. Milman, V. D. and Myskis, A. D. (1960), 'On the stability of motion in the presence of impulses'. S.mat. Zur. 1, 233 – 237(Russian).

15. Ogundiran, M.O. and Payne, V.F.(2013), 'On the Existence and Uniqueness of solution of Impulsive Quantum Stochastic Differential Equation'. Differential equations and control processes, 2.

16. Pandit, S. G. (1977), 'On Stability of impulsively perturbed differential systems'. Bull. austral: math.soc., 17 (20), 423-432.

17. Schwabik, S. (1992) Generalized ordinary differential equations. World Scientific.

18. Robert, K. (2002-03) Impulsive differential equations with non-local conditions. Morehead electronic journal of applicable mathematics. Issue 3 MATH.

19. Samojilenko, A. M. and Perestjuk, N.A. (1987), 'Differential equations with impulsive action', Golovnocizd., Kiev, (Russian).

20. Jun Liu, Xinzhi Liu, and Wei-Chau, X. (2011), 'Impulsive stabilization of stochastic functional differential equations'. Applied mathematics letters, 24, 264–269.

21. Jun Liu, Xinzhi Liu, and Wei-Chau X. (2011), 'Input-to-state stability of impulsive and switching hybrid systems with time-delay'. Automatica 47, 899–908.

22. Yang, Z, Xu, D and Xiang, L. (2006), 'Exponential p-stability of impulsive stochastic differential equations with delays'. Elsevier, 359, 129-137.